Bagrut Summer 2001, 9

Let's define $x = AB$

Given:

- $AB = DC = AD = BC = x$

- $DE = EC = \frac{DC}{2} = \frac{x}{2}$

Proof:

AE

$$\begin{aligned}
AE^2 &= AD^2 + DE^2 - 2AD \times DE \times \cos\alpha \\
&= x^2 + \left(\frac{x}{2}\right)^2 - 2x \times \frac{x}{2} \times \cos\alpha \\
&= x^2 + \frac{x^2}{4} - x^2 \cos\alpha \\
&= \frac{5x^2}{4} - x^2 \cos\alpha \\
&= x^2 \left(\frac{5}{4} - \cos\alpha\right) \\
AE &= \sqrt{x^2 \left(\frac{5}{4} - \cos\alpha\right)}
\end{aligned}$$

BE

$$\begin{aligned}
BE^2 &= BC^2 + CE^2 - 2BC \times CE \times \cos(180^\circ - \alpha) \\
&= x^2 + \left(\frac{x}{2}\right)^2 - 2x \times \frac{x}{2} \times (-\cos\alpha) \\
&= x^2 + \frac{x^2}{4} + x^2 \cos\alpha \\
&= \frac{5x^2}{4} + x^2 \cos\alpha \\
&= x^2 \left(\frac{5}{4} + \cos\alpha\right) \\
BE &= \sqrt{x^2 \left(\frac{5}{4} + \cos\alpha\right)}
\end{aligned}$$

AB = x

$$
\begin{aligned}
AB^2 &= AE^2 + BE^2 - 2 \times AE \times BE \times \cos\beta \\
&= \left(\sqrt{x^2\left(\frac{5}{4} - \cos\alpha\right)}\right)^2 + \left(\sqrt{x^2\left(\frac{5}{4} + \cos\alpha\right)}\right)^2 - 2\sqrt{x^2\left(\frac{5}{4} - \cos\alpha\right)}\sqrt{x^2\left(\frac{5}{4} + \cos\alpha\right)}\cos\beta \\
&= x^2\left(\frac{5}{4} - \cos\alpha\right) + x^2\left(\frac{5}{4} + \cos\alpha\right) - 2\sqrt{x^2\left(\frac{5}{4} - \cos\alpha\right)x^2\left(\frac{5}{4} + \cos\alpha\right)}\cos\beta \\
&= x^2\left[\left(\frac{5}{4} - \cos\alpha\right) + \left(\frac{5}{4} + \cos\alpha\right)\right] - 2\sqrt{x^4\left(\frac{5}{4} - \cos\alpha\right)\left(\frac{5}{4} + \cos\alpha\right)}\cos\beta \\
&= x^2\left(\frac{5}{4} - \cos\alpha + \frac{5}{4} + \cos\alpha\right) - 2\sqrt{x^4\left[\left(\frac{5}{4}\right)^2 - \cos^2\alpha\right]} \times \cos\beta \\
&= x^2\left(\frac{5}{4} + \frac{5}{4}\right) - 2x^2\sqrt{\left(\frac{5}{4}\right)^2 - \cos^2\alpha} \times \cos\beta \\
&= x^2\left(\frac{10}{4}\right) - 2x^2\sqrt{\frac{25}{16} - \cos^2\alpha} \times \cos\beta \\
&= \frac{10x^2}{4} - 2x^2\sqrt{\frac{25 - 16\cos^2\alpha}{16}} \times \cos\beta \\
&= x^2\left[\frac{10}{4} - 2\sqrt{\frac{1}{16} \times (25 - 16\cos^2\alpha)} \times \cos\beta\right] \\
&= x^2\left[\frac{10}{4} - 2 \times \frac{1}{4}\sqrt{25 - 16\cos^2\alpha} \times \cos\beta\right] \\
&= x^2\left[\frac{10}{4} - \frac{1}{2}\sqrt{25 - 16\cos^2\alpha} \times \cos\beta\right] \\
&= x^2\left[\frac{10}{4} - \frac{\sqrt{25 - 16\cos^2\alpha} \times \cos\beta}{2}\right] \\
x^2 &= x^2\left[\frac{10}{4} - \frac{\sqrt{25 - 16\cos^2\alpha} \times \cos\beta}{2}\right] \\
1 &= \frac{10}{4} - \frac{\sqrt{25 - 16\cos^2\alpha} \times \cos\beta}{2} \\
2 &= \frac{20}{4} - \sqrt{25 - 16\cos^2\alpha} \times \cos\beta \\
2 &= 5 - \sqrt{25 - 16\cos^2\alpha} \times \cos\beta \\
0 &= 3 - \sqrt{25 - 16\cos^2\alpha} \times \cos\beta \\
3 &= \sqrt{25 - 16\cos^2\alpha} \times \cos\beta \\
\cos\beta &= \frac{3}{\sqrt{25 - 16\cos^2\alpha}}
\end{aligned}
$$